Relativistic Eikonal Approximation for a Three-Body Scattering Amplitude.

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The eikonal technique has been successful in summing the generalized ladder type graphs for two-body scattering amplitudes (1-3), form factors (3,6), etc. Its validity for QED as opposed to φ^3 field theory has been studied (4). The bound-state poles of the two-body eikonal amplitude for QED give the Balmer formula for hydrogenlike atoms (5) and even include recoil corrections.

In view of these successes, we apply the eikonal technique to the three-body generalized ladder graphs (Fig. 1) in QED, treating the nucleus as a heavy fermion of mass M, charge Ze. Making the standard eikonal approximation, and summing over distinct generalized ladder graphs, with the factor $(l!\,m!\,n!)^{-1}$ arising from double counting (7), we find

$$(1)\qquad -i(2\pi)^4 T^{lmn} = \int \prod_{j=1}^{l} \left(\frac{\mathrm{d}r_j}{(2\pi)^4} \frac{i(4Ze^2 P\cdot q_1)}{r_j^2-\mu^2} \right) \prod_{j=1}^{m} (\dots) \prod_{j=1}^{n} (\dots) \cdot$$
$$\cdot (2\pi)^{12} \delta^4\!\left(\Delta - \sum_{j=1}^{l} r_j - \sum_{j=1}^{m} s_j \right) \delta^4\!\left(\delta_1 + \sum_{j=1}^{n} t_j - \sum_{j=1}^{l} r_j \right) \delta^4\!\left(\delta_2 - \sum_{j=1}^{m} s_j - \sum_{j=1}^{n} t_j \right) (l!\,m!\,n!)^{-1} \cdot$$
$$\cdot \sum_{P_\alpha P_\beta P_\gamma} \prod_{k=1}^{l+m-1} \frac{i}{-2P\cdot \bar{A}_k + i\varepsilon} \prod_{k=1}^{m+n-1} \frac{i}{-2q_1\cdot \bar{B}_k + i\varepsilon} \prod_{k=1}^{n+l-1} \frac{i}{-2q_2\cdot \bar{C}_k + i\varepsilon},$$

where

$$P_\alpha = \begin{pmatrix} 1 & 2 & \dots & l+m \\ a_1 & a_2 & \dots & a_{l+m} \end{pmatrix}, \qquad \bar{A}_i = \sum_{j=1}^{i} A_{a_j}$$

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and

$$\Lambda_i = \begin{cases} r_i\,, & i = 1, 2, \ldots, l\,, \\ s_{i-l}\,, & i = l + 1\,, \ldots, l + m\,, \end{cases} \tag{2}$$

and μ is the photon mass, and with similar definitions for $P_\beta, P_\gamma, \overline{B}_k, \overline{C}_k$.

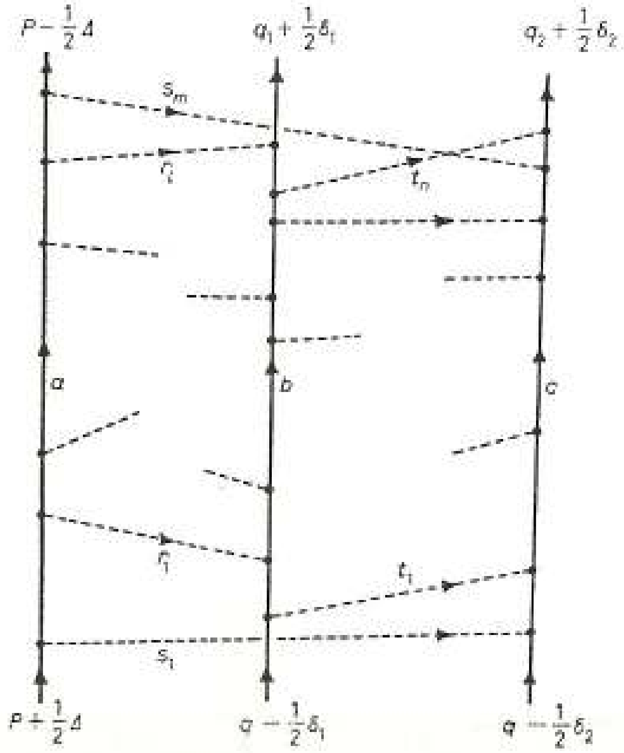

Fig. 1. – A typical generalized ladder graph involving exchange of l, m, n photons between particles (a, b), (a, c) and (b, c).

In the rest frame of a, where

$$P = (M, 0, 0, 0)\,, \qquad q_1 = (\omega_1, 0, 0, q_1) \qquad \text{and} \qquad q_2 = (\omega_2, -q_2 \sin\theta, 0, -q_2\cos\theta),$$

eq. (1) can be simplified by combining the δ-functions appropriately, and using the identity

$$\delta\left(\sum_{k=1}^{l+m} x_k\right) \sum_{P_\alpha} \prod_{k=1}^{l+m-1} \frac{1}{x_{\alpha_1} + \ldots + x_{\alpha_k} + i\varepsilon} = (-2\pi i)^{-1} \prod_{k=1}^{l+m} [-2\pi i \delta(x_k)]\,, \tag{3}$$

giving a form suitable for exponentiation. An overall δ-function $(2\pi)^4 \delta^4(\delta_1 + \delta_2 - \Delta)$ factors out and the remaining five δ-functions can be exponentiated, introducing the co-ordinates $z, \boldsymbol{b}_1, \boldsymbol{b}_2$ (the latter being vectors in the $(x\text{-}y)$-plane). Summation over $l, m, n = 1, 2, \ldots$ gives the compact form

$$-iT = \int \exp[-iz\delta_{1z} - i\boldsymbol{b}_1 \cdot \boldsymbol{\delta}_1 - i\boldsymbol{b}_2 \cdot \boldsymbol{\delta}_2]\, dz\, d^2\boldsymbol{b}_1\, d^2\boldsymbol{b}_2 \cdot \\ \cdot (8M\omega_1 q_2 \cos\theta)(\exp[-i\chi_1] - 1)(\exp[-i\chi_2] - 1)(\exp[-i\chi_{12}] - 1)\,, \tag{4}$$

where

$$\begin{aligned}
q_1\chi_1 &= 2\omega_1 Z\alpha K_0(\mu b_1)\,,\\
q_2\chi_2 &= 2\omega_2 Z\alpha K_0(\mu b_2)\,,\\
Q\chi_{12} &= -2\alpha q_1\cdot q_2 K_0(\mu\xi_{12})\,,\\
Q^2 &= \omega_2^2 q_1^2 + \omega_1^2 q_2^2\cos^2\theta + m^2 q_2^2\sin^2\theta + 2\omega_1\omega_2 q_1 q_2\cos\theta\,,\\
\xi_{12}^2 &= (b_{2y}-b_{1y})^2 + \{(b_{2x}-b_{1x})(\omega_2 q_1+\omega_1 q_2\cos\theta) - z\omega_1 q_2\sin\theta\}^2/Q^2\,.
\end{aligned}$$

To extract the bound-state poles of (4), which arise from the singularities of the integrand at $b_1 \approx 0$, $b_2 \approx 0$, we follow the procedure of ref. [3]; after putting $\cos\theta = 1$ in (4) and taking $q_1, q_2 \ll m$, since we are not interested in recoil corrections. Then the condition for bound states is given by

$$\frac{imZ\alpha}{q_1} + \frac{imZ\alpha}{q_2} - \frac{im\alpha}{q_1+q_2} = -n \qquad (n = 2, 3, \ldots)\,. \tag{5}$$

This gives for the lowest symmetrical $(1s)^2$ state ($q_1 = q_2$) the result

$$B_{1,1} = -\frac{m\alpha^2}{n^2}(2Z - \tfrac{1}{2})^2 \tag{6}$$

and for the asymmetrical excited states $1s, ns$

$$\left\{\begin{aligned}
B_{1,n} &= -\frac{mZ^2\alpha^2}{2}(1 + x_n^{-2})\,,\\
x_n &= \frac{1}{2}(n - 1 + Z^{-1}) + \frac{1}{2}\{(n-1+Z^{-1})^2 + 4n\}^{\frac{1}{2}}\,.
\end{aligned}\right. \tag{7}$$

The result (6) is compared with the simplest variational estimate (with $Z_{\text{eff}} = Z - \frac{5}{16}$), and the experimental results in Table I.

TABLE I. – *Comparison of eikonal, variational and experimental values (in electron-volt) for the ground-state energy levels of some two-electron atoms.*

Z	experimental value	variational value	eikonal value
1	− 14.35	− 12.86	− 15.3
2	− 78.98	− 77.49	− 83.32
3	− 198.04	− 196.5	− 205.7
4	− 371.5	− 370.0	− 382.5
5	− 599.4	− 597.8	− 613.7

Thus although the eikonal technique is able to sum the generalized ladder graphs for the three-body amplitude (this method can be extended to the 4-, 5-, ... body cases), it does not succeed as well as in the two-body case in predicting the bound-state energy levels. The validity of the eikonal method for the three-body amplitude in the appropriate high-energy limit may be a problem worth examining.